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Differential Equations

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First Exam

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1. Find the solution to the following IVP and determine all possible behaviors of the solution as  $t \rightarrow 0$  from right. If this behavior depends on the value of  $y_0$  give this dependence. (14 pt's.)

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$$ty' + 2y = 4t^2, \quad y(1) = y_0.$$

$$\Rightarrow y' + \frac{2}{t}y = 4t$$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = e^{\ln t^2} = t^2$$

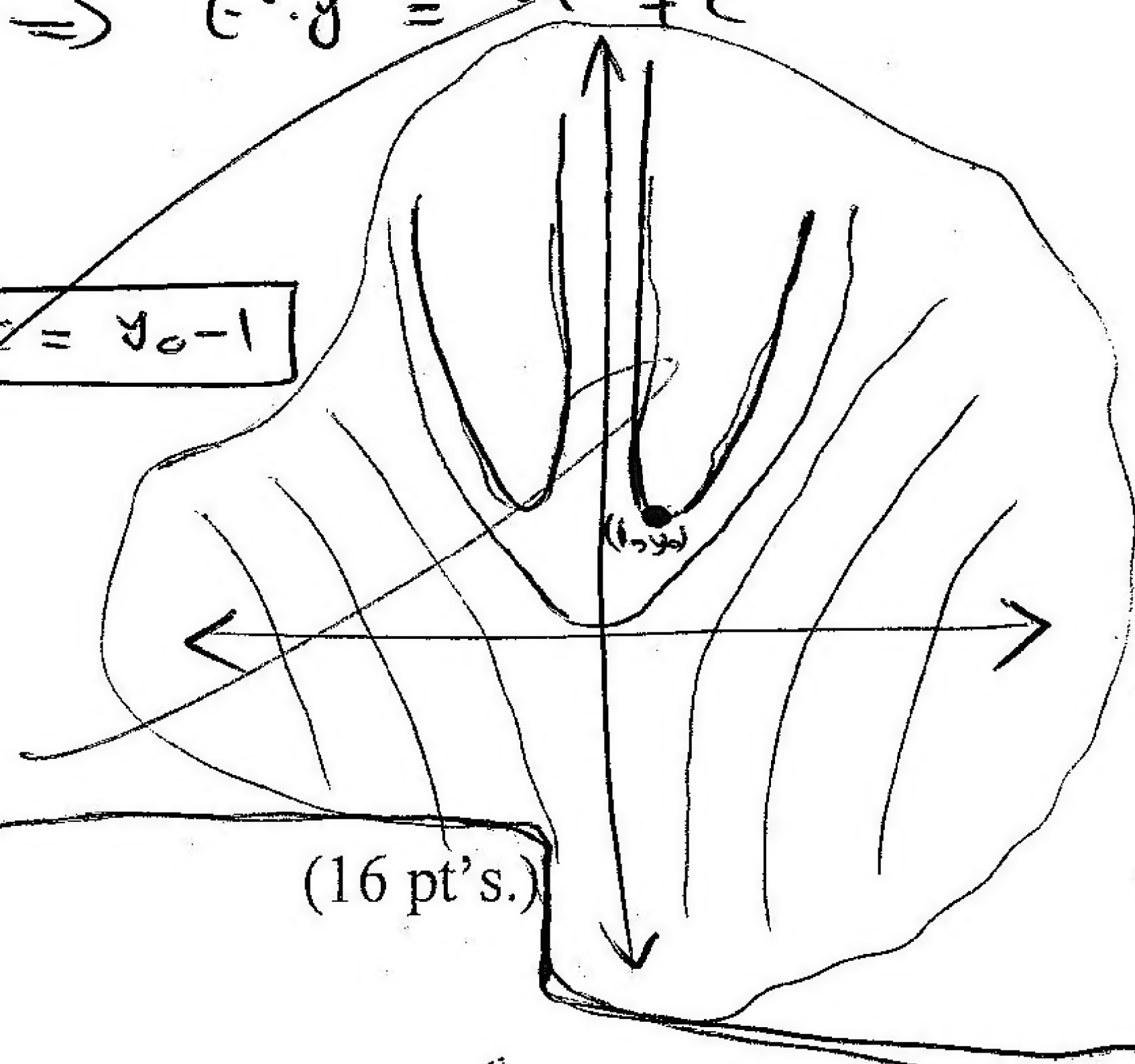
$$\therefore t^2 y' + 2ty = 4t^3 \Rightarrow \frac{d}{dt} [t^2 y] = 4t^3$$

$$\Rightarrow \therefore t^2 y = \int 4t^3 dt \Rightarrow t^2 y = \frac{4t^4}{4} + C \Rightarrow t^2 y = t^4 + C$$

$$\Rightarrow \therefore y = t^2 + \frac{C}{t^2}$$

$$y(1) = y_0 \Rightarrow y_0 = 1^2 + \frac{C}{1^2} \Rightarrow y_0 = 1 + C = 0 \Rightarrow C = y_0 - 1$$

$$\text{as } t \rightarrow 0 \Rightarrow y \rightarrow +\infty$$



2. Solve the following DE's.

(16 pt's.)

a.  $\frac{dy}{dx} + \frac{1}{x}y = xy^2$

$$y' + \frac{1}{x}y = xy^2 \times \frac{1}{y^2} \Rightarrow y' y^{-2} + \frac{1}{x} y^{-1} = x$$

$$\Rightarrow \text{let } v = y^{-1} \Rightarrow v' = -1 y^{-2} y'$$

$$\therefore -v' + \frac{1}{x}v = x \Rightarrow \cancel{v' - \frac{1}{x}v = x} \Rightarrow v' - \frac{1}{x}v = -x$$

$$\cancel{\frac{dv}{dx} + \frac{1}{x}v = x} \Rightarrow \frac{dv}{dx} = \frac{1}{x}v - x$$

$$\mu(t) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$\therefore v' \cdot \frac{1}{x} + \frac{1}{x^2}v = -1 \Rightarrow \frac{d}{dx} \left[ \frac{1}{x} \cdot v \right] = -1$$

$$\Rightarrow \left[ \frac{1}{x} \cdot v \right] = \int -1 dx \Rightarrow \frac{v}{x} = -x + C \Rightarrow v = -x^2 + Cx$$

$$\text{but } v = y^{-1}$$

$$\therefore y = \frac{1}{-x^2 + Cx}$$

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$y' = \frac{3y^2 - x^2}{2xy}$   
 $y = vx \Rightarrow y' = v + x \frac{dv}{dx}$

$-(2xy) + (3y^2 - x^2) \frac{dy}{dx} = 0$

(13 pt's.)

(2)

b.  $\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$

$2xy \frac{dy}{dx} = 3y^2 - x^2$

$\Rightarrow$  Exact D.E.

we can solve

it like that

$M_y = 3y^2 - x^2$

$\Rightarrow M_y = 6y$

$N_x = 2xy$

$\Rightarrow N_x = 2y$

$\rightarrow$  Not exact

$\mu(t) = e^{\int \frac{M_y - N_x}{N} dx} = \frac{6y - 2y}{2xy} = \frac{4y}{2xy} = \frac{2}{x}$

$\Rightarrow \mu(x) = x^2$

and Solve

c.  $xydx + (2x^2 + 3y^2 - 20)dy = 0$

(17 pt's.)

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$(xy) + (2x^2 + 3y^2 - 20) \frac{dy}{dx} = 0$

$M_y = x$   
 $N_x = 4x$

$\rightarrow$  Not exact

Integrating factor

$\mu(y) = \frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}$

$\frac{d\mu}{dy} = \frac{N_x - M_y}{M} \cdot \mu \Rightarrow \frac{d\mu}{\mu} = \frac{3}{y} dy$

$\Rightarrow \mu(y) = e^{\int \frac{3}{y} dy} = e^{3 \ln y} = e^{\ln y^3} = y^3$

$(xy^4) + (2x^2y^3 + 3y^5 - 20y^3) \frac{dy}{dx} = 0$

$\Rightarrow M_y = 4xy^3$   
 $N_x = 4xy^3$

$\rightarrow$  exact

$\psi(x,y) = \int M_x dx + h(y)$

$\psi(x,y) = \int (xy^4) dx + h(y) = \frac{1}{2} x^2 y^4 + h(y)$

$\psi_y = 2x^2 y^3 + h'(y) = 2x^2 y^3 + 3y^5 - 20y^3$

$\psi_y = 2x^2 y^3 + h'(y) = 2x^2 y^3 + 3y^5 - 20y^3$

Exact Method / another method

$\psi(x,y) = \int M_x dx + \int n(y) dy$

$n(y)$ : the terms of  $N_x$  that doesn't contain  $x$

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$\therefore \psi(x,y) = \int 4xy^3 dx + \int (3y^5 - 20y^3) dy$

$= \frac{4x^2 y^3}{2} + \frac{3y^6}{6} - \frac{20y^4}{4} = 2x^2 y^3 + \frac{y^6}{2} - 5y^4 = C$



3. a. Solve the IVP  $y'' - 2y' + 5y = 0$ ,  $y(\frac{\pi}{2}) = 0$ ,  $y'(\frac{\pi}{2}) = 2$ .

(8 pt's.)

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$$r^2 - 2r + 5 = 0$$

$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$r = 1 + 2i$$

$$r = 1 - 2i$$

$$\therefore y = e^{\lambda t} [c_1 \cos \mu t + c_2 \sin \mu t] = e^t [c_1 \cos 2t + c_2 \sin 2t]$$

$$y(\frac{\pi}{2}) = 0 \Rightarrow c_1 e^{\frac{\pi}{2}} \cos \pi + c_2 e^{\frac{\pi}{2}} \sin \pi = 0 \Rightarrow -c_1 e^{\frac{\pi}{2}} = 0$$

$$\Rightarrow c_1 = 0$$

$$y = c_1 [2e^t (-\sin 2t) + \cos 2t e^t] + c_2 [2e^t \cos 2t + \sin 2t (e^t)]$$

$$y'(\frac{\pi}{2}) = c_1 [2e^{\frac{\pi}{2}} (-\sin \pi) + \cos \pi e^{\frac{\pi}{2}}] + c_2 [2e^{\frac{\pi}{2}} \cos \pi + \sin \pi (e^{\frac{\pi}{2}})]$$

$$= c_1 [0 + -e^{\frac{\pi}{2}}] + c_2 [-2e^{\frac{\pi}{2}} + 0] = 2$$

$$-2c_2 e^{\frac{\pi}{2}} = 2 \Rightarrow c_2 e^{\frac{\pi}{2}} = -1 \Rightarrow c_2 = \frac{-1}{e^{\frac{\pi}{2}}} = -e^{-\frac{\pi}{2}}$$

$$\therefore y = -e^{-\frac{\pi}{2}} \cdot e^t \sin 2t$$

$$(-2) \frac{e^{-2+1}}{-2+1} \cdot 2 = \frac{2e^{-1}}{-1} = \frac{-2}{t}$$

b. If  $y_1$  and  $y_2$  are two linearly independent solutions of  $t^2 y'' - 2y' + (3+t)y = 0$  and if  $W(y_1, y_2)(2) = 3$ , find the value of  $W(y_1, y_2)(4)$ .

(10 pt's.)

(10)

$$y'' - \frac{2}{t^2} y' + \frac{3+t}{t^2} y = 0$$

$$W = ce^{-\int P(t) dt} = e^{-\int -\frac{2}{t^2} dt} = e^{\frac{2}{t}} = e^{2t^{-1}} = e^{\frac{2}{t}}$$

$$\Rightarrow W(y_1, y_2)(2) = e^{\frac{2}{2}} = 3 \Rightarrow e = 3 \Rightarrow \ln e = \ln 3$$

$$\Rightarrow -1 + c = \ln 3 \Rightarrow c = \ln 3 + 1$$

$$\therefore W(y_1, y_2)(4) = e^{\frac{2}{4} + \ln 3 + 1} = e^{\frac{1}{2} + \ln 3} = e^{\frac{1}{2}} \cdot e^{\ln 3} = 3e^{\frac{1}{2}}$$

c. Find the longest interval in which the IVP  $t(t-4)y'' + 3ty' + 4y = 2$ ,  $y(3) = 0$ ,  $y'(3) = -1$  is certain to have a unique solution. Do not attempt to find the solution.

(6 pt's.)

(6)

$$y'' + \frac{3t}{t(t-4)} y' + \frac{4}{t(t-4)} y = \frac{2}{t(t-4)}$$

$$\Rightarrow t(t-4) = 0 \Rightarrow t = 0, t = 4$$

$$(-\infty, 0) \cup (0, 4) \cup (4, \infty)$$

The longest interval is

$$0 < t < 4$$

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3. If the function  $y_1 = x^2$  is a solution of  $x^2 y'' - 3xy' + 4y = 0$ . Find the general solution of the DE on  $(0, \infty)$ . (16 pt's.)

$$y = x^2 v$$

$$y' = x^2 v' + (2x)v$$

$$y'' = x^2 v'' + v'(2x) + (2x)v' + v(2) = x^2 v'' + 4xv' + 2v$$

$$\therefore x^2(x^2 v'' + 4xv' + 2v) - 3x(x^2 v' + 2xv) + 4(x^2 v) = 0$$

$$(x^4 v'' + 4x^3 v' + 2x^2 v) - (3x^3 v' + 6x^2 v) + 4x^2 v = 0$$

$$v''(x^4) + v'(4x^3 - 3x^3) + v(2x^2 - 6x^2 + 4x^2) = 0$$

$$v''(x^4) + v'(x^3) + \cancel{v(-4x^2)} = 0$$

$$\therefore x^4 v'' + x^3 v' + \cancel{12x^2 v} = 0$$

$$v'' + \frac{1}{x} v' + \cancel{12x^2 v} = 0$$

$$\text{let } w = v' \Rightarrow w' = v''$$

$$w' + \frac{1}{x} w = 0$$

$$\mu(x) = e^{\int \frac{1}{x}} = e^{\ln x} = x$$

$$x w' + \frac{1}{x} \cdot x w = 0$$

$$\frac{d}{dx} [w \cdot x] = 0$$

$$w \cdot x = c \Rightarrow w = \frac{c}{x}$$

$$\text{but } w = v' \Rightarrow v' = \frac{c}{x} \Rightarrow v = \int \frac{c}{x} = c \ln(x) + d$$

$$\therefore v = c \ln(x) + d$$

$$\text{but } y = x^2 v \Rightarrow y = c x^2 \ln x + \boxed{x^2 d} \rightarrow \text{repeated}$$